University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

Problems 42-47 are devoted to multipole radiation, bremsstrahlung, synchrotron radiation, and radiation from a free-electron laser.

42.
Working in the far zone $r' \ll \lambda \ll r$, consider azimuthally symmetric ($m = 0$) electric quadrupole ($\mathrm{E}2_0$) radiation. At a particular angular frequency ω, work with the complex fields $\vec{\tilde{B}}(\vec{r})$ and $\vec{\tilde{E}}(\vec{r})$ defined by

$$\vec{B}(\vec{r},t) \equiv \mathrm{Re}\left(\vec{\tilde{B}}(\vec{r})e^{-i\omega t}\right)$$
$$\vec{E}(\vec{r},t) \equiv \mathrm{Re}\left(\vec{\tilde{E}}(\vec{r})e^{-i\omega t}\right) .$$

For E-type radiation, the magnetic field $\vec{\tilde{B}}$ ($\perp \hat{r}$) is proportional to the vector spherical harmonic $\vec{X}$:

$$\vec{\tilde{B}} \propto \vec{X}_{20}(\theta,\phi) \equiv \vec{L}\,Y_{20}(\theta,\phi) ,$$

with $i\vec{L} \equiv \vec{r} \times \nabla$. Use the fact that

$$\vec{\tilde{E}} \approx c\vec{\tilde{B}} \times \hat{r}$$

in the far zone. Obtain a function $f(\theta,\phi)$ such that the radiated power P in the far zone is proportional to it:

$$\frac{dP}{d\Omega} \propto f(\theta,\phi) .$$

43.
At $t = 0$, charges $+e$ lie on the top right and bottom left corners of a square of side b in the xy plane that is centered at the origin; charges $-e$ lie on the top left and bottom right corners.
(a.)
Determine the lowest-l nonvanishing electrostatic multipole moment(s) of the charge distribution.
(b.)
The static charge distribution in (a.) now is set into oscillation: as time advances, the position vector of each charge is multiplied by the same factor $1 + \epsilon\cos\omega t$, where ω and $0 < \epsilon \ll 1$ are real constants. (Note that the charges do not move in a circle.) Using the fact that a static electric multipole corresponding to a given l and m, when caused to oscillate, yields E-type (TM) multipole radiation of the *same* l and m, what type(s) of radiation (*e.g.* $\mathrm{E}1_0$) is (are) emitted?
(c.)
Using the facts introduced in the previous problem, but generalizing them to the spherical harmonic(s) appropriate here, obtain a function $f(\theta,\phi)$ such that the radiated power P in the far zone $b \ll \frac{2\pi c}{\omega} \ll r$ is proportional to it:

$$\frac{dP}{d\Omega} \propto f(\theta,\phi) .$$

At how many points on the unit sphere (*e.g.* the north pole) does this radiation pattern vanish?

44.
Griffiths Problem 11.15.

45.
Start from the expression derived in class for the energy radiated by an accelerating point charge per steradian per unit of *retarded* time t':

$$\frac{dW}{d\Omega\,dt'} = \Big(\frac{q}{4\pi\epsilon_0}\Big)^2\frac{\epsilon_0}{c}\,\frac{\big|\hat{R}\times\big[(\hat{R}-\vec{\beta})\times\dot{\vec{\beta}}\big]\big|^2}{(1-\hat{R}\cdot\vec{\beta})^5} .$$

Consider *synchrotron radiation* by a particle of charge q moving in a circular orbit of radius b in a coordinate system where

$$\hat{\beta} = \hat{z}$$
$$\hat{\dot{\beta}} = \hat{x} ,$$

i.e. $\hat{x}$ points toward the center of the circle and $\hat{z}$ points along its circumference in the particle's direction of motion. Define

$$\hat{R} \equiv (n_x, n_y, n_z) ,$$

where $\hat{n}$ is a unit vector extending from the particle in an arbitrary direction towards which an element of radiation is emitted.

(a.)

Show that

$$\hat{R} \times \left[(\hat{R} - \vec{\beta}) \times \hat{\dot{\beta}} \right] = \hat{n} n_x - \hat{x} - \beta \hat{n} \times \hat{y} \; .$$

(b.)

Using this result, show that

$$\left| \hat{R} \times \left[(\hat{R} - \vec{\beta}) \times \hat{\dot{\beta}} \right] \right|^2 = 1 - 2\beta n_z + \beta^2 n_z^2 - (1-\beta^2) n_x^2$$

(c.)

Consider a set of spherical polar coordinates centered at the particle (*not* at the center of the beam circle). Taking θ to be the polar angle of $\hat{n}$ relative to $\hat{z}$, and ϕ to be its azimuth about $\hat{z}$, express n_x and n_z in terms of θ and ϕ.

(d.)

Using the results of (b.) and (c.), show that

$$\frac{dW}{d\Omega\, dt'} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{\epsilon_0}{c} \times$$
$$\times \frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3} \left(1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2 (1-\beta\cos\theta)^2} \right) .$$

46.

Consider the result of the previous problem in the relativistic limit $\gamma \gg 1$. In that limit, the nonnegligible part of the total radiation that is emitted occurs at polar angles θ such that $\gamma\theta$ is of order unity.

(a.)

Approximating $\cos\theta$ and $\sin\theta$ to lowest nonvanishing order in θ, perform the integration over $d\Omega = d(\cos\theta)\, d\phi$, integrating by parts where necessary, to show that

$$4\pi\epsilon_0 \frac{dW}{dt'} = \frac{2}{3c^3} (q\dot{\beta}c)^2 \gamma^4 \; .$$

[Note that $(q\dot{\beta}c)^2$ is equivalent to $\ddot{p}^2$, where p is the electric dipole moment of the point charge relative to the origin. Therefore this result is the same as the (nonrelativistic) Larmor formula, except for the additional factor γ^4.]

(b.)

In terms of the |momentum| P of the point charge and its rest mass m, show that

$$4\pi\epsilon_0 \frac{dW}{dt'} = \frac{2q^2}{3c^3} \frac{P^4}{m^4 b^2} \; ,$$

and thus that the power lost to synchrotron radiation depends on the fourth power of P, the inverse fourth power of m (making it usually negligible for all but electrons), and the inverse square of b.

(c.)

Suppose that you use an electron synchrotron that taxpayers can afford. It circulates highly relativistic electrons with $\beta \approx 1$. You want to build a new synchrotron with the same beam current, the same power lost to synchrotron radiation, but twice the beam momentum. Show that the radius b of the new synchrotron must increase by a factor of 16.

47.

A free-electron laser consists of a beam of electrons (with constant velocity βc) passing through a structure known as a *wiggler* or *undulator*. (These structures are used also in sections of a circular electron synchrotron such as the ALS.) Take the beam direction to be $\hat{z}$. Consider an alternating set of magnets (for compactness, these are often permanent magnets, made of samarium cobalt as developed at LBL by the late Klaus Halbach). With a full period Δz, they produce a strong magnetic field that points alternately in the $+\hat{x}$ and $-\hat{x}$ directions.

(a.)

In the rest frame $\mathcal{S}'$ of the electron, with what fundamental angular frequency ω' does the magnetic field from the wiggler appear to oscillate?

(b.)

In $\mathcal{S}'$, the oscillating electron produces electromagnetic radiation with angular frequency ω'. Applying the relativistic Doppler shift to ("forward") radiation emitted along the beam direction, what angular frequency ω does that radiation have in the laboratory frame?

(c.)

Express λ, the wavelength of the forward radiation, as a multiple of Δz.

(d.)

At LBL's ALS, using an alternating set of magnets with $\Delta z = 10$ cm, an experimenter wishes to study the effect upon condensed-matter samples of a soft X-ray beam of wavelength 5 nm. Use this information to estimate the ALS beam energy (in GeV).